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METHOD OF FINITE ELEMENTS FOR SOLVING SOME HEAT-CONDUCTION PROBLEMS

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Heat-conduction problems are investigated with the aid of a new variational method, namely, the method of finite elements (MFE).

Heat processes in constructions or in mechanical equipment with high-temperature gradients are at present studied in a number of investigations. For a theoretical solution of the problems thus arising one can employ, in principle, either analytic or numerical methods. The classical analytic methods, including the methods based on integral transformations, can produce satisfactory solutions for simple physical models; their use, however, in involved problems, in practice, is rather doubtful. The numerical methods employed until recently were almost exclusively based on the method of finite differences. Variational methods have at present found wide application (in particular, the MFE), since the use of the latter results in matrix equations suitable for processing on digital computers.

The main concept of the MFE consists in subdividing the entire solution domain into a set of a finite number of elements, the links between adjacent elements being provided in a finite number only of the so-called points of contact. The continuous solution of the original problem in the old domain (for heat conduction the latter is the temperature field) is replaced by a piecewise polynomial one with values specified in advance at the nodes of the complex. Since these values are the same for adjacent elements therefore continuity of the solution is attained in the entire domain under investigation. Some of the main advantages of the MFE are the ease of satisfying any boundary conditions for bodies of quite different shapes including holes and complicated boundaries and also that any inhomogeneities or anisotropy can be taken into account, and finally that one can solve nonlinear problems with the aid of various iteration procedures.

The unsteady heat-conduction equation can be written as follows:

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[\lambda_{ij}(x,\ t,\ \theta)\frac{\partial\theta}{\partial x_j} - \rho(x,\ t,\ \theta)\,c(x,\ t,\ \theta)\,\dot{\theta} + Q(x,\ t,\ \theta) = 0,\right. \tag{1}$$

where $\dot{\theta} = \partial\theta/\partial t$ with the boundary and initial conditions

$$\theta(x,\ 0) = \theta^0(x),\ \frac{\partial\theta}{\partial n} + \alpha(x,\ t,\ \theta)(\theta - \theta_m)|_{A^\alpha} = 0,$$
$$\theta|_{A^\theta} = \theta^A(x,\ t),\quad \frac{\partial\theta}{\partial n} + q_p(x,\ t,\ \theta)|_{A^q} = 0, \tag{2}$$

where $A = A^\theta \cup A^q \cup A^\alpha$ is the boundary of the domain Ω under investigation; $\partial\theta/\partial n$ is the normal derivative to the surface and α is the heat-emission coefficient. Of the variational methods employed in heat problems the Galerkin method is the one most often used [1]. One considers the basic space of the functions φ such that

$$\varphi \in H \text{ and } \varphi|_{A^\theta} = 0,$$

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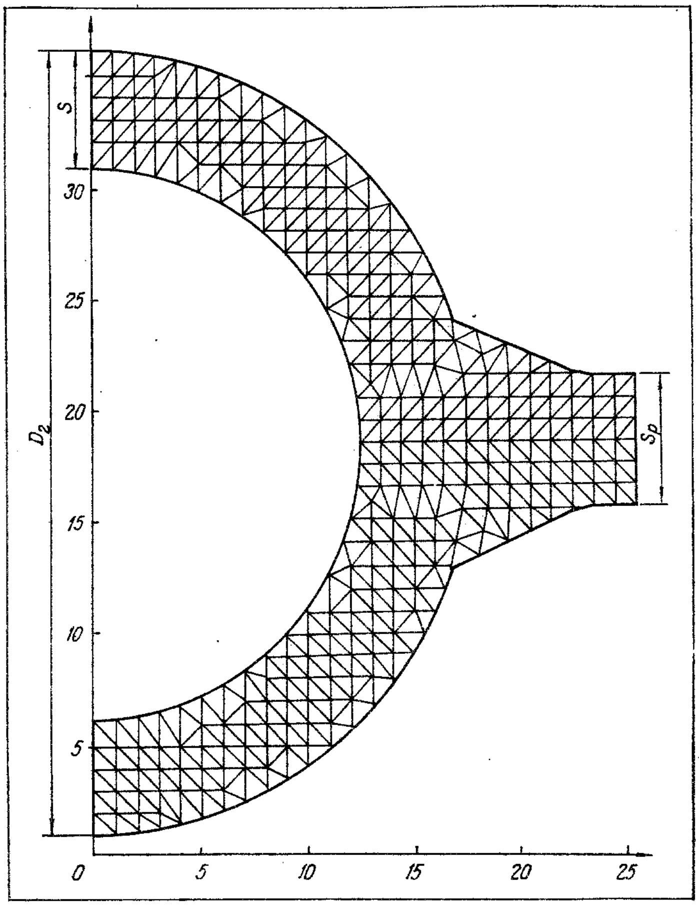

Fig. 1. Grid for evaluation of the temperature field of a membrane wall: D_2 = 35 mm; S_p = 5 mm; S = 6 mm.

where H is the Hilbert space with the scalar product

$$(u,\ \varphi)=\int\limits_{\Omega} u\varphi dx.$$

Equation (1) is now multiplied by the function φ and integrated over the domain Ω. By using Green's theorem one finds

$$\int\limits_{\Omega}\rho c\dot{\theta}\varphi dV+\int\limits_{\Omega}\sum_{i,j=1}^{N}\lambda_{ij}\frac{\partial\theta}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dV+\int\limits_{A^{\alpha}}\alpha\theta\varphi dA=\int\limits_{\Omega}Q\varphi dV+\int\limits_{A^{\alpha}}\alpha\theta_m\varphi dA+\int\limits_{A^{q}}q^{0}\varphi dA. \quad (3)$$

If one introduces the notation

$$k(u,\ \varphi)=\int\limits_{\Omega}\sum_{i,j=1}^{N}\lambda_{ij}\frac{\partial u}{\partial x_i}\frac{\partial\varphi}{\partial x_j}dV+\int\limits_{A^{\alpha}}\alpha u\varphi dA,$$
$$p(\varphi)=\int\limits_{\Omega}Q\varphi dV+\int\limits_{A^{\alpha}}\alpha\theta_m\varphi dA+\int\limits_{A^{q}}q^{0}\varphi dA, \quad (4)$$

one can write (3) as

$$(\rho c\dot{\theta},\ \varphi)+k(\theta,\ \varphi)=p(\varphi). \quad (5)$$

A function of θ which satisfies the initial condition and the integral identity (5) is a weak solution of the problem (1) and (2).

The Galerkin method suitable for the heat-conduction problem can be formulated as follows: The approximate solution θ_h is sought in the form

$$\theta_h=\Sigma a_i(t)\varphi_i(x). \quad (6)$$

The time-continuous Galerkin solution is found from a discrete analog of Eq. (5). If (6) is inserted into (5) then the basic MFE equation is obtained for the solution of the heat problems; its matrix form is as follows:

$$[C]\{\dot{a}\} + [H]\{a\} = \{F\}, \tag{7}$$

where the entries of the matrices [C], [H], and of the vector {F} are given by

$$c_{ij} = (\rho c \varphi_i,\ \varphi_j),\ h_{ij} = k(\varphi_i,\ \varphi_j),\ f_i = p(\varphi_i). \tag{8}$$

If the coefficients λ, ρ, c, α of the heat-conduction equation, the internal heat sources, and the given heat flux on the boundary of the domain under investigation depend on the temperature, then the matrices H and C as well as the vector F are also functions of the temperature, and (7) represents a system of nonlinear differential equations. The linearization method is used on each time interval Δt. It was proposed by Comini in [2] that the so-called three-layer scheme be used (previously introduced by Lels in [3]). If it is assumed that the temperature changes linearly on the small interval (t − Δt, t + Δt), then Eq. (7) can be approximated as follows:

$$[C(t)]\frac{(\{a(t+\Delta t)\} - \{a(t-\Delta t)\})}{2\Delta t} + [H(t)]\frac{(\{a(t+\Delta t)\} + \{a(t)\} + a(t-\Delta t)\})}{3} = \{F(t)\}, \tag{9}$$

and after algebraic transformation one obtains

$$([H(t)] + \frac{3}{2\Delta t}[C(t)])\{a(t+\Delta t)\} = \frac{3}{2\Delta t}[C(t)] \times \{a(t-\Delta t)\} - 3\{F(t)\} - [H(t)](\{a(t)\} + \{a(t-\Delta t)\}). \tag{10}$$

In the system of linear algebraic equations thus obtained the matrices of the system and the vector on the right must be reevaluated at each temporal step.

The problem of radiative heat transfer can be given as another example which results in a linear system of equations. The heat flux at the boundary can be described by the condition

$$\frac{\partial\theta}{\partial n} + \varepsilon\sigma(\theta^4 - \theta_z^4)|_{A^z} = 0, \tag{11}$$

where $\sigma = 5.67 \cdot 10^{-8}\ W \cdot m^{-2} \cdot {}^\circ K^{-4}$ is the Stefan–Boltzmann constant and ε is "the mutual radiation coefficient." If this condition is incorporated in Eq. (3) then a nonlinear system of equations

$$[C]\{\dot{a}\} + [H]\{a\} + [Z]\{a^4\} = \{F\} \tag{12}$$

is obtained, whose numerical solution is very difficult to find if the system is large. It was proved that it is convenient to represent the condition (11) in the form

$$\frac{\partial\theta}{\partial n} + \beta(\theta - \theta_z)|_{A^z} = 0, \tag{13}$$

where

$$\beta = \varepsilon\sigma(\theta + \theta_z)(\theta^2 + \theta_z^2)$$

is similar in form to the third boundary condition in (2). The "constant" β is determined in the interval (t, t + Δt) with the aid of the temperatures evaluated at the instant t; by solving the system of equations (7) the temperatures are determined at the nodes of the grid at the instant t + Δt. To improve the accuracy, the computation can be repeated several times, each time with a corrected value of β; the results, however, differ so little that it is enough to repeat the computation only once and thus shorten the machine time.

In practice, one also has to solve numerically the system of equations (7), which becomes a system of ordinary differential equations after the above-described changes have been carried out. To solve the latter, explicit methods can be used. However, they have many disadvantages since the matrix [C] is not diagonal. The majority of the classical approximation methods for solving ordinary differential equations result in relatively stable schemes. To be able to solve the system (7) with any temporal step, an absolutely stable method is necessary. A single step procedure is adopted whose scheme for solving the differential equations $\dot{a} = g(a, t)$, $a(0) = a^0$ is given by

$$a^{n+1} - a^n = \Delta t\,[(1 - \vartheta)\,g^{n+1} + \vartheta g^n]. \tag{14}$$

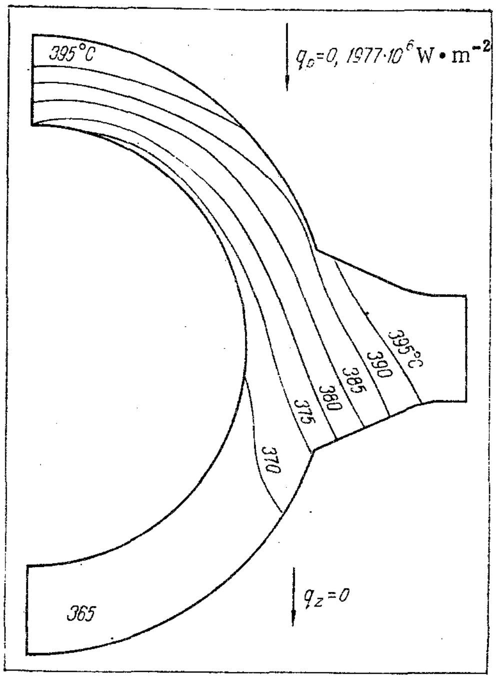

Fig. 2. Isotherms obtained from computations of temperatures at grid nodes. $\theta_m = 367°C$; $\alpha = 0.043 \cdot 10^6$ W·m^{-2}·°K^{-1}.

A necessary and sufficient condition for the scheme (14) to be stable is $\vartheta \leqslant 1/2$. For $\varphi = 1/2$ the method is often referred to as the trapezoid method; by applying it to Eq. (7) one obtains the well-known Crank—Nichols procedure

$$\left([C]+\frac{1}{2}\Delta t\,[H]\right)\{a(t)\}=\left([C]-\frac{1}{2}\Delta t\,[H]\right)\{a(t-\Delta t)\}+\frac{1}{2}\Delta t\,(\{F(t)\}+\{F(t-\Delta t)\}). \tag{15}$$

Another approach to find a solution is to use the space—time elements. The Galerkin method can be used again in Eq. (7) since the vectors $\{a\}$ and $\{F\}$ can be approximated in the interval Δt by values taken at several selected points of this interval. The error in the results can then be reduced by using a high-degree interpolation method.

To provide an illustration for the above, the temperature field is determined of a section of a tubular system where inside the tube the heat of forced convection is emitted, and also a radiation flux falls on part of its outside. The computations were carried out using the program ROTER on the EC 1030 computer compiled for solving a two-dimensional MFE problem with a selected triangular element with three nodes. The temperature curve was approximated by a linear polynomial on each element. In this case the effect of radiation was replaced by a given heat flux $q_p = 0.1977 \cdot 10^6$ W·m^{-2} and its value along the surface was expressed by means of a product of the angular coefficient of radiation and the heat flux q_p. In Fig. 1 the grid is shown used for the computation as well as the input parameters. From the computed temperature at the nodes of the grid the isotherms were constructed (Fig. 2) which provide a clear map of temperature distribution in the domain. In the lower part of the tube the temperature grows slowly (from 365° to 370°C) since the outside is thermally insulated.

On the basis of the carried-out analysis and the above given example one can draw the conclusion that the application of the MFE method is useful for solving stationary or non-stationary heat-conduction problems in bodies of virtually any geometrical form in the case of realistic boundary and initial conditions and when the thermophysical parameters of the bodies can depend on temperature or other parameters.

NOTATION

θ, temperature; θ_m, ambient temperature; Q, inner source intensity; θ^A, prescribed temperature on the boundary; λ, coefficient of thermal conductivity; α, heat-transfer coefficient; ρ, density; c, heat capacity; q_p, prescribed heat flux; σ, Stefan—Boltzmann constant; ε, mutual radiation coefficient.

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